The diameter variability of the product graphs

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Abstract

The diameter of a graph can be affected by the addition or the deletion of some edges. In [3], we have studied the diameter variability of the Cartesian product of graphs. In this paper we discuss about two fundamental products, strong and lexicographic products of graphs, whose diameter increases (decreases) by the deletion (addition) of a single edge. The problems of minimality and maximality of the product graphs with respect to its diameter are also solved. These problems are motivated by the fact that these graph products are good interconnection networks.

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1 Introduction

An interconnection network connects the processors of a parallel and distributed system. The topological structure of an interconnection network can be modeled by a connected graph where the vertices represent sites of the network and the edges represent communication links. The diameter is

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often taken as a measure of efficiency, especially for networks with maximum time - delay or signal degradation. If some links are faulty, the information cannot be transmitted by these links and the efficiency of network will be affected. In fact, many graph products are good interconnection networks and a good network must be hard to disrupt and the transmissions must remain connected even if some vertices or edges fail [7].

Let $G = (V, E)$ be a simple connected graph with $|V| = n$ and $|E| = m$. The *distance* between u and v in G, $d(u, v)$ is the length of a shortest path joining them in G. The *diamete*r of a graph G, diam(G) is the maximum distance between any two vertices in G. For a vertex $u \in V(G)$, if there exists a vertex $v \in V(G)$ such that $d(u, v) = \text{diam}(G)$, v is then called a diametral vertex of u. For all notions not given here, see [12].

The *strong product* of two graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1 = u_2$ and $v_1 - v_2 \in E(H)$ or $u_1 - u_2 \in E(G)$ and $v_1 = v_2$ or $u_1 - u_2 \in E(G)$ and $v_1 - v_2 \in E(H)$. Also, $\text{diam}(G \boxtimes H) = max\{diam(G), diam(H)\}$ [8].

The *lexicographic product* of two graphs G and H, denoted by $G \circ H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices $(u_1, v_1), (u_2, v_2)$ are adjacent if either $u_1 - u_2 \in E(G)$ or $u_1 = u_2$ and $v_1 - v_2 \in E(H)$. If $G \neq K_n$, then $\text{diam}(G \circ H) = \text{diam}(G)$ and $\text{diam}(K_n \circ H) = 2$ [8].

The diameter of a graph may increase or decrease due to the addition or the deletion of some edges. The following notations are used to denote the *diameter variability* [11] of a graph G. Let $k \geqslant 1$ be an integer.

$D^k(G)$: the minimum number of edges to be deleted from G to increase the diameter of G by (at least) k.

$D^{-k}(G)$: the minimum number of edges to be added to G to decrease the diameter of G by (at least) k.

A graph G is diameter minimal if $\text{diam}(G - e) > \text{diam}(G)$ for any $e \in E(G)$ and is diameter maximal if $\text{diam}(G + e) < d(G)$ for any $e \notin E(G)$ [2].

In [3], we have studied the diameter variability of the Cartesian product of graphs. In [4], chithra studied the diameter variability of a Mycielski graph. In [11], J. J. Wang et al. studied the diameter variability of cycles and tori. In [6], Graham and Harary studied the diameter variability of hypercubes. In [1], Bouabdallah et al. improved the lower bound of $D^0(Q_n)$ and gave an upper bound. Some notions related to diameter variability already studied are diameter vulnerability and fault diameter. The problem of determining diameter vulnerability and fault diameter was proposed by Chung and Garey [5], Krishnamoorthy and Krishnamurthy [9] respectively. More studies can be referred in [10].

The diameter of a graph plays a significant role in analyzing the efficiency of an interconnection network. The diameter is often taken as a measure of efficiency, when studying the potential effects of link failures on the performance of a communication network, especially for networks with maximum time-delay or signal degradation. In fact, most of the graph products are interconnection networks and a good network must be hard to disrupt and the transmissions must remain connected even if some vertices or edges fail. Thus, the notion of diameter variability has great applications in networks. This motivated us to study the diameter variability of the strong and lexicographic product of graphs. The notions of diameter minimality and diameter maximality of the product graphs are also studied. An upper bound for $D^1(G)$ is also obtained.

Here, we consider only connected graphs H_1, H_2 and denote the $V(H_1) = \{u_1, u_2, ..., u_{n_1}\}$, $V(H_2) = \{v_1, v_2, ..., v_{n_2}\}$ and $V(H_1 \boxtimes H_2) = V(H_1 \circ H_2) = \{u_1v_1, u_1v_2, ..., u_{n_1}v_{n_2}\}$. Also, $|E(H_1)| = m_1$ and $|E(H_2)| = m_2$. Since, $H_1 \boxtimes K_1 \cong H_1$ and $H_1 \circ K_1 \cong H_1$ we assume that H_1, $H_2 \neq K_1$.

2 Diameter variability of the strong product of graphs

Theorem 2.1. *Let $G \cong H_1 \boxtimes H_2$. Then $D^1(G) = 1$ if and only if G is any one of the following graphs where,*
(a) both H_1 and H_2 are complete graphs.
(b) H_1 and H_2 are not complete graphs with diam(H_1) = diam(H_2) and either H_1 or H_2 have at least one pair of vertices with exactly one diametral path or there exists an edge in H_1 or H_2 that is on all diametral paths between any two vertices.

Proof. Let $G \cong K_{n_1} \boxtimes K_{n_2}$ where $n_1, n_2 \geqslant 2$. Then G is a complete graph and the deletion of any edge increases the diam(G).

Let H_1 and H_2 be not complete graphs with diam(H_1) = diam(H_2) and either H_1 or H_2 have at least one pair of vertices with exactly one diametral path or there exists an edge in H_1 or H_2 that is on all diametral paths between any two vertices. Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$. Consider a pair of diametral vertices u_xv_w, u_yv_z in G, by a path $u_xv_w - u_{x+1}v_{w+1} - u_{x+2}v_{w+2}$ ... $u_{y-1}v_{z-1} - u_yv_z$. Let an edge $u_xv_w - u_{x+1}v_{w+1}$, be deleted. Then, $d(u_xv_w, u_yv_z) = \text{diam}(G) + 1$ by a path $u_xv_w - u_xv_{w+1} - u_{x+1}v_{w+1} - ... - u_{y-1}v_{z-1} - u_yv_z$, where $d(u_xv_w, u_{x+1}v_{w+1}) = 2$, $d(u_{x+1}v_{w+1}, u_yv_z) = \text{diam}(G) - 1$.

Conversely suppose that $D^1(G) = 1$.

Suppose that H_1 is a not complete graph and H_2 is a complete graph.

Let an edge $u_iv_p - u_iv_q$ or $u_iv_p - u_jv_p$ or $u_iv_p - u_jv_{p+1}$, be deleted. Then $d(u_iv_p, u_iv_q) = d(u_iv_p, u_jv_p) = d(u_iv_p, u_jv_{p+1}) = 2$ by the paths $u_iv_p - u_{i+1}v_q - u_iv_q$, $u_iv_p - u_jv_{p+1} - u_jv_p$ and $u_iv_p - u_iv_{p+1} - u_jv_{p+1}$ respectively. Also, the distance between any two other vertices is not affected by the removal of this edge. Thus, when one factor is a complete graph and the other factor is a not complete graph, a minimum of two edges should be deleted to increase the diam(G). Hence, both the factors should be complete. This proves (a).

Suppose that H_1 and H_2 are not complete graphs with $\text{diam}(H_1) > \text{diam}(H_2)$.
Consider a pair of diametral vertices u_xv_w, u_yv_z in G by a path $u_xv_w - u_{x+1}v_{w+1} - u_{x+2}v_{w+2}$ ... $u_{y-1}v_{z-1} - u_yv_z$. Let an edge $u_xv_w - u_{x+1}v_{w+1}$, be deleted. Then, $d(u_xv_w, u_yv_z) = \text{diam}(H_2) + 1$ by a path $u_xv_w - u_xv_{w+1} - u_{x+1}v_{w+1}$ ... $u_{y-1}v_{z-1} - u_yv_z$, where $d(u_xv_w, u_{x+1}v_{w+1}) = 2$, $d(u_{x+1}v_{w+1}, u_yv_z) = \text{diam}(H_2) - 1$.
Hence, the $\text{diam}(G)$ remains the same. Thus, when H_1 and H_2 are not complete graphs with different diameter, at least two edges should be deleted to increase the $\text{diam}(G)$.

Suppose that H_1 and H_2 are not complete graphs with $\text{diam}(H_1) = \text{diam}(H_2)$.
Consider a pair of diametral vertices u_xv_w, u_yv_z in G. Since, $\text{diam}(H_1) = \text{diam}(H_2)$, $u_xv_w - u_{x+1}v_{w+1} - u_{x+2}v_{w+2}$ ... $u_{y-1}v_{z-1} - u_yv_z$ is a shortest path between them in G. Then, the deletion of an edge $u_iv_j - u_{i+1}v_{j+1}$ from this path increases the $\text{diam}(G)$ only if either there exists only one diametral path between u_x, u_y in H_1 and v_w, v_z in H_2 or $u_i - u_{i+1}$ is an edge in H_1 that is on all diametral paths between any two vertices in H_1 and $v_j - v_{j+1}$ is an edge in H_2 that is on all diametral paths between any two vertices in H_2. Otherwise, there exists an alternative path of length

diam(H_1) between u_xv_w, u_yv_z in G. Hence, H_1 and H_2 are not complete graphs with diam(H_1) = diam(H_2) and either H_1 or H_2 have at least one pair of vertices with exactly one diametral path or there exists an edge in H_1 or H_2 that is on all diametral paths between any two vertices. This proves (b). □

Corollary 2.2. *$G \cong H_1 \boxtimes H_2$ is diameter minimal if and only if both H_1 and H_2 are complete graphs.*

Theorem 2.3. *Let $G \cong H_1 \boxtimes H_2$. Then $D^1(G) \leqslant P(1+\delta(H_2))$, where P is the minimum number of edge disjoint paths of length diam(H_1) between any two vertices in H_1.*

Proof. Let u_x and u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$. Consider a pair of diametral vertices u_xv_z and u_yv_z in G. Let the edges $u_xv_z - u_qv_z$, $u_xv_z - u_qv_r$, where u_qs are the vertices adjacent to u_x in H_1 and v_rs are the vertices adjacent to v_z in H_2, be deleted. Then, $d(u_xv_z, u_yv_z) = \text{diam}(G) + 1$ by a path $u_xv_z - u_xv_{z+1} - u_{x+1}v_z - ... - u_{y-1}v_z - u_yv_z$ where $d(u_{x+1}v_z, u_yv_z) = \text{diam}(G) - 1$, $d(u_xv_z, u_{x+1}v_z) = 2$. Also, $d(u_xv_z, u_qv_z) = 2$ and $d(u_xv_z, u_qv_r) = 2$, since there are paths of length two between them.
Thus, $D^1(G) \leqslant P(1+\delta(H_2))$. □

Theorem 2.4. *Let $G \cong H_1 \boxtimes H_2$ be connected graph. Then $D^{-1}(G) = 1$ if and only if H_2 has a universal vertex and H_1 is a connected graph with diam(H_1) $\geqslant 4$ and $D^{-2}(H_1) = 1$ if an edge is added between a diametral vertex and any other vertex of H_1 and $D^{-1}(H_1) = 1$ if an edge is added between any two other vertices of H_1.*

Proof. Let $G \cong H_1 \boxtimes H_2$ and diam(G) = diam(H_1).

Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$. Suppose that v_1 is a universal vertex of H_2.

Let $D^{-1}(H_1) = 1$, where diam(H_1) $\geqslant 4$.

Consider a pair of diametral vertices u_xv_w, u_yv_z in G. Let an edge $u_pv_1 - u_qv_1$, where $u_p \neq u_x$, $u_q \neq u_y$, be added in G. Then, $d(u_xv_w, u_yv_z)$ $= \text{diam}(G) - 1$ by a path $u_xv_w - u_{x+1}v_1 - u_{x+2}v_1 \; ... \; u_{y-1}v_1 - u_yv_z$ where $d(u_xv_w, u_{x+1}v_1) = 1$, $d(u_{x+1}v_1, u_{y-1}v_1) = \text{diam}(G) - 3$ and $d(u_{y-1}v_1, u_yv_z) = 1$.

Consider a pair of diametral vertices u_xv_w, u_yv_z in G. Let an edge $u_xv_1 - u_yv_1$, be added in G. Then, $d(u_xv_w, u_yv_z) = 3$ by a path $u_xv_w - u_xv_1 - u_yv_1 - u_yv_z$.

Suppose that $D^{-2}(H_1) = 1$, where $\text{diam}(H_1) \geqslant 4$.

Consider a pair of diametral vertices u_xv_w, u_yv_z in G. Let an edge $u_xv_1 - u_iv_1$, where u_i is a vertex in a diamertal path between u_x and u_y in H_1, be added in G. Then, $d(u_xv_w, u_yv_z) = \text{diam}(G) - 1$ by a path $u_xv_w - u_xv_1 - u_iv_1 - ... - u_{y-1}v_1 - u_yv_z$ where $d(u_xv_w, u_xv_1) = 1$, $d(u_xv_1, u_{y-1}v_1) = \text{diam}(G) - 3$ and $d(u_{y-1}v_1, u_yv_z) = 1$. Thus, the distance between any two vertices in G is at most diam(G)-1.

Conversely suppose that $D^{-1}(G) = 1$. If both H_1 and H_2 are complete graphs, then G is a complete graph. If $\text{diam}(H_1) = 2$, then the addition of a single edge in G will not make G a complete graph. Also, if $\text{diam}(H_1) = 3$, then the addition of a single edge in G will not decrease the $\text{diam}(G)$, since there exists a path of length at least three between any pair of diametral vertices in G. Thus, it is clear that H_1 is a connected graph with $\text{diam}(H_1) \geqslant 4$.

Suppose that H_1 is any connected graph and H_2 is any connected graph without a universal vertex.

Let v_p and v_q be a pair of non adjacent vertices in H_2. Consider a pair of diametral vertices u_xv_q, u_yv_q in G. Let an edge $u_iv_p - u_jv_p$, be added in G. Since v_p is not adjacent to v_q, the diametral path between u_xv_q and u_yv_q does not contain the edge $u_iv_p - u_jv_p$ in G. Hence, to decrease the diam(G), H_2 should contain a universal vertex.

Suppose that H_2 has a universal vertex v_1. Consider a pair of diametral vertices u_xv_w, u_yv_w in G. Let an edge $u_iv_1 - u_jv_1$, be added in G.

Let $i \neq x$, $j \neq y$.
Consider a diametral path $u_xv_w - u_{x+1}v_1 - u_{x+2}v_1 - ... - u_{y-1}v_1 - u_yv_w$ between u_xv_w, u_yv_w in G. Then, $d(u_xv_w, u_{x+1}v_1) = 1$ and $d(u_{y-1}v_1, u_yv_w) = 1$, since H_2 has a universal vertex. Now, consider the distance between the remaining vertices in the diametral path. Then, the diam(G) decreases by one only if
$d(u_{x+2}v_1, u_{y-1}v_1) =$[diam($H_1$)-2]-1 = diam($H_1$)-3. Hence, to decrease the diam(G) by one, the distance between u_xv_1 and u_yv_1 should be decreased by one, by the addition of a single edge.

Let $i = x$, $j = y$.
Then, $d(u_xv_w, u_yv_w) = 3$ by a path $u_xv_w - u_xv_1 - u_yv_1 - u_yv_w$, since H_2 has a universal vertex. From the previous case it follows that diam(G) decreases, only if $d(u_pv_1, u_qv_1) \leqslant \text{diam}(H_1) - 1$. Hence, to decrease the diam(G) by one, the distance between u_xv_1 and u_yv_1 should be decreased by one, by the addition of a single edge.

Now, let $i = x$, $j \neq y$.
Consider a diametral path $u_xv_w - u_xv_1 - u_{x+1}v_1 - \ \ldots\ - u_{y-1}v_1 - u_yv_w$ between u_xv_w, u_yv_w in G. Then $d(u_xv_w, u_xv_1) = 1$ and $d(u_{y-1}v_1, u_yv_w) = 1$, since H_2 has a universal vertex. Now, consider the distance between the remaining vertices in the diametral path. Then, the diam(G) decreases by one, only if $d(u_xv_1, u_{y-1}v_1)$ =[diam(H_1)-1]-2 = diam(H_1)-3. Hence, to decrease the diam(G) by one, the distance between u_xv_1 and $u_{y-1}v_1$ should be decreased by two, by the addition of a single edge. □

Corollary 2.5. *There does not exist a graph $G \cong H_1 \boxtimes H_2$ such that G is diameter maximal.*

Proof. In Theorem 2.4 we have characterized the strong product of graphs whose diameter decreases by the addition of a single edge. Hence, we need to prove the theorem only for such Gs.

Suppose that H_2 is a not complete graph with a universal vertex and H_1 is a connected graph with $D^{-1}(H_1) = 1$ or $D^{-2}(H_1) = 1$ with $\text{diam}(H_1) \geqslant 4$. Let an edge $u_xv_p - u_xv_q$ be added in G, then the diam(G) remains the same, since $\text{diam}(G) = \text{diam}(H_1)$.

Suppose that H_2 is a complete graph and H_1 is a connected graph with $D^{-1}(H_1) = 1$ or $D^{-2}(H_1) = 1$ with $\text{diam}(H_1) \geqslant 4$. Let the three vertices u_x, u_s and u_r form a P_3 in H_1. Consider a pair of diametral vertices u_xv_p, u_yv_p in G. Let an edge $u_xv_q - u_rv_p$ where v_q is a neighbour of v_p in H_2, be added. Then the addition of an edge $u_xv_q - u_rv_p$ does not decrease the distance between them in G. Thus, $d(u_xv_p, u_yv_p) = \text{diam}(G)$. Hence, there exists some $e \notin E(G)$ such that $\text{diam}(G + e) = \text{diam}(G)$. □

3 Diameter variability of the lexicographic product of graphs

Theorem 3.1. *Let $G \cong H_1 \circ H_2$. Then $D^1(G) = 1$ if and only if G is any one of the following graphs where,*
(a) both H_1 and H_2 are complete graphs.

(b) $H_1 = K_2$ or a connected graph with diameter two in which there exists at least one pair of adjacent vertices with no path of length two between them and H_2 is a disconnected graph in which there exists at least one component with an isolated vertex.

Proof. (a) Let $G \cong K_{n_1} \circ K_{n_2}$, where $n_1, n_2 \geqslant 2$. Then, the deletion of any edge increases the diam(G).

(b) Suppose that $H_1 = K_2$ and H_2 is a disconnected graph with an isolated vertex v_p, then diam(G)=2. Let an edge $u_iv_p - u_jv_p$, be deleted. There is a path $u_iv_p - u_jv_q - u_iv_q - u_jv_p$ of length three between them.

Let H_1 be a connected graph with diameter two in which the adjacent vertices u_r, u_s have no path of length two between them and H_2 be a disconnected graph with an isolated vertex v_p, then diam(G) = 2. Let an edge $u_rv_p - u_sv_p$, be deleted. There is a path $u_rv_p - u_sv_q - u_rv_q - u_sv_p$ of length three between them.

Conversely suppose that $D^1(G) = 1$.
Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$.

Suppose that H_1 is a complete graph and H_2 is any connected graph, then diam$(G) \leqslant 2$.

Let an edge $u_iv_p - u_iv_q$ or $u_iv_p - u_jv_p$ or $u_iv_p - u_jv_q$, be deleted. There exists at least two paths of length two between these pairs of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus to increase the diam(G) by one, H_2 should be a complete graph. This proves (a).

Suppose that H_1 is a connected graph.

Let an edge $u_iv_w - u_jv_w$, be deleted. If H_2 is any connected graph, then there exists at least $\kappa(H_2) + 1$ paths $u_xv_w - u_{x+1}v_z \ ... \ u_{y-1}v_z - u_yv_w$ of length diam(H_1) between u_xv_w and u_yv_w in G, where $z \in \{1, 2, \ ... \ , n_2\}$. Thus, when H_2 is a connected graph, at least two edges should be deleted to increase the diam(G). Hence, it is clear that H_2 should be a disconnected graph.

Now, if H_2 is a disconnected graph without an isolated vertex, then there exists at least two paths of length diam(G) between a pair of diametral vertices $u_x v_w$ and $u_y v_w$ in G. Thus, at least two edges should be deleted to increase the diam(G). Hence, H_2 is a disconnected graph in which there exists at least one component with an isolated vertex.

If diam$(H_1) \geqslant 3$, then the deletion of an edge will not increase the diam(G). There is a path of length at most three between each pair of vertices. Hence, H_1 is any connected graph with diam$(H_1) \leqslant 2$.

Let H_1 be a complete graph with $n_1 > 2$.

Since $n_1 > 2$ there exists at least two paths of length two between each pair of vertices in G. Thus, the deletion of an edge from G does not increase the diam(G). Hence, $n_1 = 2$.

Let diam$(H_1) = 2$.

Let an edge $u_i v_p - u_j v_p$, be deleted. Then the diam(G) increases only if u_i and u_j have no path of length two between them in H_1. Otherwise, at least two edges should be deleted to increase the diam(G). Also, the distance between any two other vertices is not affected by the removal of these edges. Hence, H_1 should be a connected graph with diameter two in which there exists at least one pair of adjacent vertices with no path of length two between them.
This proves (b). □

Corollary 3.2. *$G \cong H_1 \circ H_2$ is diameter minimal if and only if G is any one of the following graphs where,*
(a) both H_1 and H_2 are complete graphs.
(b) $H_1 = K_2$ or a connected graph with diameter two in which there is no path of length two between any two adjacent vertices in H_1 and H_2 is a totally disconnected graph.

Proof. (a) Let $G = K_{n_1} \circ K_{n_2}$. Then, G is diameter minimal.

(b) Suppose that H_1 is a K_2 and H_2 is a totally disconnected graph, then diam$(G) = 2$. Let an edge $u_i v_p - u_j v_p$ or $u_i v_p - u_j v_q$, be deleted. Then, there is a path $u_i v_p - u_j v_q - u_i v_q - u_j v_p$ or $u_i v_p - u_j v_p - u_i v_q - u_j v_q$ of length three between each pair of vertices. Thus, the deletion of any edge increases the diam(G).

Suppose that H_1 is a connected graph with diameter two in which there is no path of length two between any two adjacent vertices in H_1 and H_2 is a totally disconnected graph, then diam(G) $= 2$. Let an edge $u_i v_p - u_j v_p$

or $u_iv_p - u_jv_q$, be deleted. There is a path of length three between these pairs of vertices. Thus, the deletion of any edge increases the diam(G).

Hence, G is diameter minimal.

Conversely suppose that G is diameter minimal. In Theorem 3.3 we have characterized the lexicographic product of graphs whose diameter increases by the deletion of a single edge. Hence, we need to prove the theorem only for such Gs.

Let $G \cong K_{n_1} \circ K_{n_2}$. Then, clearly G is diameter minimal.

Suppose that $H_1 = K_2$ and H_2 is a disconnected graph in which there exists at least one component with an isolated vertex.

Let an edge $u_iv_p - u_iv_q$ where v_p, v_q are not isolated vertices in H_2, be deleted. Since v_p, v_q are not isolated vertices there is a path of length two between u_iv_p and u_iv_q in G. Hence, if H_2 contains any pair of adjacent vertices, the deletion of that edge will not increase the diam(G). Thus, H_2 is a totally disconnected graph.

Suppose that H_1 is a connected graph with diameter two in which at least one pair of adjacent vertices have no path of length two between them and H_2 is a disconnected graph in which there exists at least one component with an isolated vertex.

As in the previous case, if H_2 contains any pair of adjacent vertices, the deletion of that edge will not increase the diam(G). Hence, H_2 is a totally disconnected graph.

Let an edge $u_iv_p - u_jv_p$ where the adjacent vertices u_i and u_j have a path of length two in H_1, be deleted. If any two adjacent vertices in H_1 have a path of length two between them, then the deletion of an edge will not increase the diam(G). Thus, H_1 is a connected graph with diameter two in which there is no path of length two between any two adjacent vertices in H_1. □

Theorem 3.3. *Let $G \cong H_1 \circ H_2$. Then $D^1(G) \leqslant t\ n_2$, where t is the minimum number of edge disjoint paths of length diam(H_1) between any two vertices in H_1.*

Proof. Follows from Theorem 2.3. □

Theorem 3.4. *Let $G \cong H_1 \circ H_2$. Then $D^{-1}(G) = 1$ if and only if G is any one of the following graphs where,*
(a) H_2 has a universal vertex and H_1 is a connected graph with $diam(H_1) \geqslant 4$ and $D^{-2}(H_1) = 1$ if an edge is added between a diametral vertex and any other vertex of H_1.
(b) H_2 is any graph and H_1 is a connected graph with $diam(H_1) \geqslant 4$ and $D^{-1}(H_1) = 1$ if an edge is added between the diametral vertices or between any two other vertices of H_1.

Proof. Follows from Theorem 2.4. □

Corollary 3.5. *There does not exist a graph $G \cong H_1 \circ H_2$ such that G is diameter maximal.*

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References

[1] A. Bouabdallah, C. Delorme, S. Djelloul , Edge deletion preserving the diameter of the hypercube, Discrete Appl. Math. 63 (1995), 91-95.

[2] F. Buckley, F. Harary, Distance in graphs, Addison - Wesley Publishing Company (1990).

[3] Chithra M.R., A. Vijayakumar, The Diameter variability of the Cartesian product of graphs, Discrete Math. Algorithms Appl. Vol. 6 (2014), 1450001-1450009.

[4] Chithra M.R., Changing and unchanging the diameter of a Mycielski graphs, Ars Combinatoria (to appear).

[5] F.R.K. Chung, M.R. Garey, Diameter bounds for altered graphs, J. Graph Theory 8 (4) (1984), 511-534.

[6] N. Graham, F. Harary, Changing and unchanging the diameter of a hypercube, Discrete Appl. Math. 37/38 (1992), 265-274.

[7] L. H. Hsu, C. K. Lin, Graph theory and interconnection networks, CRC Press (2009).

[8] W. Imrich, S. Klavžar, R. Hammack, Handbook of Product Graphs, CRC Press (2011).

[9] M.S. Krishnamoorthy, B. Krishnamurthy, Fault diameter of interconnection networks, Comput. Math. Appl. 13 (1987), 577-582.

[10] Revathy A.S., Regitha R Nair, Chithra M.R., A Survey on - How the diameter of a graph is affected by the removal and the addition of some edges, Int. J. Appl. Eng. Res Vol. 10, No 16 (2015) , 36169-36174.

[11] J. J. Wang, T. Y. Ho, D. Ferrero, T. Y. Sung, Diameter variability of cycles and tori, Inform. Sci. 178 (2008), 2960 - 2967.

[12] D. B. West, Introduction to Graph Theory, Prentice Hall of India (2003).